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WHEN IS IT ALLOWED TO TREAT THE HEAT-TRANSFER COEFFICIENT α AS CONSTANT?

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Abstract—The problem here is discussed when the heat-transfer coefficient α may be treated as constant. By means of example of the slug-flow in a finite length (in flow direction) flat duct with prescribed heat flux on the boundary the criterial moduli for the problem are derived and the ranges of their numerical values are determined.

NOMENCLATURE

a, duct half-width;
d_t, equivalent thermal diameter in Jordan sense [1], (for the flat duct $d_t = 2.a$);
$f(x/L)$, $= f(\xi)$, arbitrary, continuous function describing the heat flux distribution along the duct walls;
$f'(\xi)$, $= \dfrac{d}{d\xi}[f(\xi)]$;
L, length of the heated (or cooled) duct-sector;
$\bar{L}$, $= L/a$;
n, $= 0, 1, 2, \ldots$, successive integer number;
q_{max}, maximal value of the heat flux for $0 \leqslant x \leqslant L$;
$\bar{q}$, $= q_{max}.r_t/\lambda.t_i$;
r_t, $= d_t/2$, (for the flat duct $r_t = a$);
t, fluid temperature;
w, fluid velocity;
x, dimensional coordinate parallel to the flow direction;
y, dimensional coordinate normal to the duct walls.

Greek letters

α, heat transfer coefficient at the duct wall;
η, $= y/a$;
ϑ, $= (t-t_i)/t_i$, dimensionless fluid temperature;
κ, molecular thermal diffusivity of the fluid;
λ, molecular thermal conductivity of the fluid;
ξ, $= x/L$;
π, $= 3{\cdot}141593\ldots$.

Subscripts

i, initial value (inlet value) of the fluid temperature;
m, mean value of the fluid temperature (bulk temperature);
w, fluid temperature at the duct wall (for $y = a$ or $\eta = 1$).

Dimensionless groups

$\left.\begin{array}{l} 2.\bar{q}, \\ 2.\bar{L}/Pe, \end{array}\right\}$ criteria of admission of the assumption $\alpha = \text{const}$;
Pe, $= w.d_t/\kappa = Re.Pr$, Peclét number;
Pr, Prandtl number;
Re, Reynolds number.

SINCE the first pioneer-works of Graetz (1883, 1885) a great amount of research work has been performed in the field of the forced convection in ducts with various shapes of their cross-section. Many of these works, in particular the newer ones, were cited in the excellent book of Kays [2]. The enormous contribution to the knowledge on the forced convection in liquid metals has been gathered in the course of many years due to the research effort of Dwyer and his Group [3–15]. A further illustration of the great research effort of investigators in many countries is shown in [16].

However, nowhere up to now, according to the author's recognition, the question has been put, in what situations for finite length ducts the assumptions usually made about the heat-transfer coefficient α are admissible. It has been demonstrated in [2] that for fully developed flow with constant heat flux along the duct walls the heat-transfer coefficient α may be treated as constant and then:

$$t_w - t_m = \text{const}. \tag{1}$$

From this assumption it follows immediately that:

$$\frac{dt_w}{dx} - \frac{dt_m}{dx} = 0 \tag{2}$$

and:

$$\frac{\partial t}{\partial x} = \frac{dt_w}{dx} = \frac{dt_m}{dx}. \tag{3}$$

The above assumption introduced to the energy equation causes the change of its type from parabolic to elliptic and makes its solution considerably easier. The assumption (1) and its consequences (2) and (3) were applied by Axford [17], Dwyer [8–14], Nijsing and Eifler [18] as well as by Ushakov *et al.* [19]. Strictly speaking, the problem of forced convection in ducts should be considered in the conjugated formulation, i.e. when the energy equations for the duct and for the ambient medium are solved simultaneously.

Now the question can be posed whether the equation (2) holds always true for constant heat flux along the finite length duct wall, and, if not, under what hydraulic-thermal conditions the equation (2) may be treated as satisfied.

Therefore, it seems to be needed to create some criterion, which would make it possible to estimate the validity of equation (2). Such criterion should include: the geometrical parameters of the duct such as the length of the heated (or cooled) duct sector and the thermal radius of duct cross-section, as well as the hydraulic-thermal parameters: Reynolds and Prandtl (or Peclét) numbers.

Up to now, in all theoretical works concerning the generalized Graetz-problem the dimensionless coordinate in flow direction has been usually expressed in the form $\xi = (x/r_t)/(Re.Pr) = (x/r_t)/Pe$. In such definition the finite length of the duct does not appear distinctly.

The problem [4–7] after use of dimensionless coordinates defined in Nomenclature takes the form:

$$\frac{\partial^2 \vartheta}{\partial \eta^2} - \frac{Pe}{2.T} \cdot \frac{\partial \vartheta}{\partial \xi} = 0, \tag{8}$$

$$\vartheta(\eta, 0) = 0, \tag{9}$$

$$\left.\frac{\partial \vartheta}{\partial \eta}\right|_{\eta=0} = 0, \tag{10}$$

$$\left.\frac{\partial \vartheta}{\partial \eta}\right|_{\eta=1} = \bar{q}.f(\xi). \tag{11}$$

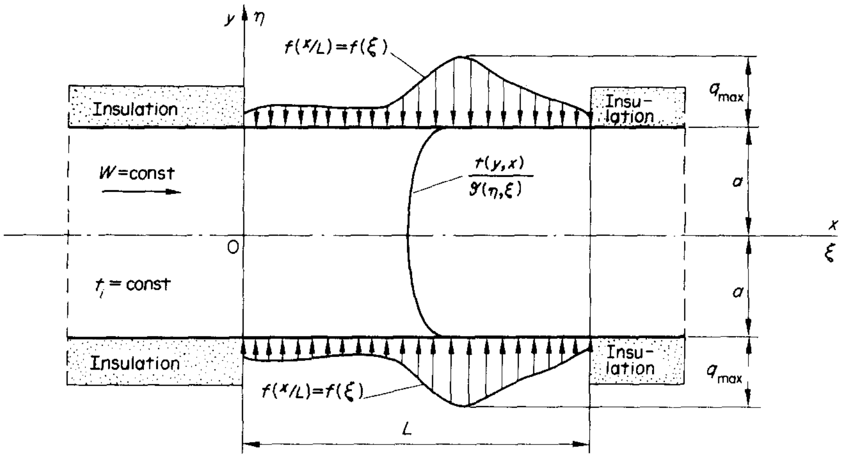

FIG. 1. Model of the slug flow in a finite length flat duct with an arbitrary heat flux distribution along the bounding walls.

Let us try to give an answer to the title question in the form of some criterion taking into consideration the example of the slug-flow between two infinitely thin plates with arbitrary (in flow direction) heat flux distribution. The slug flow through the flat duct has, therefore, been chosen, because in such cases the eigenfunctions and the eigenvalues of the appropriate Sturm–Liouville problem can be most easily determined. All considerations presented can be transferred onto more geometrically complicated ducts as well as on the laminar or turbulent flow.

The two-dimensional temperature field in the fluid flowing through the duct shown in Fig. 1 is described by the energy equation in the following form:

$$\frac{\partial^2 t}{\partial y^2} - \frac{w}{\kappa} \cdot \frac{\partial t}{\partial x} = 0, \tag{4}$$

where accordingly to the considerations of Hsu [20] the axial conductivity of the fluid was neglected.

The solution of this equation should satisfy the initial (inlet) condition:

$$t(y, 0) = t_i = \text{const} \tag{5}$$

and the boundary conditions:

$$\lambda . \left.\frac{\partial t}{\partial y}\right|_{y=0} = 0 \tag{6}$$

$$\lambda . \left.\frac{\partial t}{\partial y}\right|_{y=a} = q_{\max} \cdot f(x/L). \tag{7}$$

The solution of the above problem was obtained in [21] by use of the variables separation and is expressed as follows:

$$\vartheta(\eta, \xi) = \bar{q} . \left[\frac{2.\bar{L}}{Pe} \cdot \int_0^{\xi} f(\xi').\mathrm{d}\xi' + \left(\frac{\eta^2}{2} - \tfrac{1}{6}\right).f(\xi) \right.$$
$$+ 2 . \sum_{n=1}^{\infty} (-1)^n . \cos(n.\pi.\eta)$$
$$\times \left\{ \frac{2.\bar{L}}{Pe} \int_0^{\xi} \exp\left[-\frac{2\,\bar{L}}{Pe}.(n.\pi)^2.(\xi - \xi') \right] \right.$$
$$\left.\left. \times f(\xi').\mathrm{d}\xi' + -\frac{f(\xi)}{(n.\pi)^2} \right\} \right], \tag{12}$$

where, for the reason of simple duct-geometry, constant velocity of the fluid and the absence of the function describing the distribution of the eddy diffusivity of heat, the eigenfunctions of the appropriate Sturm–Liouville problem are expressed as the infinite cosine-sequence $\{\cos(n.\pi.\eta)\}$ and the eigenvalues are the successive multiples of π-number.

One can notice at once, that the solution in the form (12) satisfies the boundary conditions (10) and (11). The initial condition (9) is also satisfied, because:

$$\frac{\eta^2}{2} - \tfrac{1}{6} = 2 . \sum_{n=1}^{\infty} (-1)^n . \frac{\cos(n.\pi.\eta)}{(n.\pi)^2}. \tag{13}$$

Inserting (13) into (12) one obtains:

$$\vartheta(\eta,\xi)=\bar{q}.\left[\frac{2.\bar{L}}{Pe}.\int_0^{\xi} f(\xi').\mathrm{d}\xi' +2.\sum_{n=1}^{\infty}(-1)^n.\cos(n.\pi.\eta) \times\left\{\frac{2.\bar{L}}{Pe}.\int_0^{\xi}\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2 \times(\xi-\xi')\right].f(\xi').\mathrm{d}\xi'\right\}\right]. \quad (14)$$

The above expression is the second equivalent form of the solution satisfying exactly the initial condition (9).

The solution (14) is, except the criterial numbers, identical to the relation (40) given in [22]. This relation represents the solution of the unsteady heat conduction problem in a plate with the time-dependent Neumann boundary condition. Such problem is the mathematical analogue of the problem considered (8–11).

Integrating by parts the integrals appearing under the summation sign in relations (12) and (14) one can present these relations in the form easier for further considerations. Namely:

$$\vartheta(\eta,\xi)=\bar{q}.\left[\frac{2.\bar{L}}{Pe}.\int_0^{\xi} f(\xi').\mathrm{d}\xi'+\left(\frac{\eta^2}{2}-\frac{1}{6}\right).f(\xi) +2.\sum_{n=1}^{\infty}(-1)^n.\frac{\cos(n.\pi.\eta)}{(n.\pi)^2} \times\left\{\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.\xi\right].f(0) +\int_0^{\xi}\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.(\xi-\xi')\right] \times f'(\xi').\mathrm{d}\xi'\right\}\right] \quad (12a)$$

and:

$$\vartheta(\eta,\xi)=\bar{q}.\left[\frac{2.\bar{L}}{Pe}.\int_0^{\xi} f(\xi').\mathrm{d}\xi' +2.\sum_{n=1}^{\infty}(-1)^n.\frac{\cos(n.\pi.\eta)}{(n.\pi)^2}.\left\{f(\xi) -\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.\xi\right].f(0) -\int_0^{\xi}\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.(\xi-\xi')\right] \times f'(\xi').\mathrm{d}\xi'\right\}\right]. \quad (14a)$$

The calculation of the bulk temperature by use of the expressions (12), (14), (12a) or (14a) leads to the formula:

$$\vartheta_m(\xi)=\frac{\int_0^1\vartheta(\eta,\xi).\mathrm{d}\eta}{\int_0^1\mathrm{d}\eta}=\bar{q}.\frac{2.\bar{L}}{Pe}.\int_0^{\xi}f(\xi').\mathrm{d}\xi'. \quad (15)$$

The relation (2) expressed in dimensionless form is:

$$\frac{\mathrm{d}\vartheta_w}{\mathrm{d}\xi}-\frac{\mathrm{d}\vartheta_m^?}{\mathrm{d}\xi}=0. \quad (2a)$$

Using the above relations one can verify to what degree the assumed equality (2) or (2a) is satisfied. Namely:

$$\frac{\mathrm{d}\vartheta_w}{\mathrm{d}\xi}=\left.\frac{\partial\vartheta}{\partial\xi}\right|_{\eta=1} =\bar{q}.\frac{2.\bar{L}}{Pe}.\left[f(\xi)+\tfrac{1}{3}.f'(\xi) +2.\sum_{n=1}^{\infty}\left\{\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.\xi\right].f(0) +\int_0^{\xi}\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.(\xi-\xi')\right] \times f'(\xi').\mathrm{d}\xi'-\frac{f'(\xi)}{(n.\pi)^2}\right\}\right] \quad (16)$$

and:

$$\frac{\mathrm{d}\vartheta_m}{\mathrm{d}\xi}=\bar{q}.\frac{2.\bar{L}}{Pe}.f(\xi). \quad (17)$$

Taking (13) into consideration one can express (16) in the following form:

$$\frac{\mathrm{d}\vartheta_w}{\mathrm{d}\xi} =\bar{q}.\frac{2.\bar{L}}{Pe}.\left[f(\xi)+2.\sum_{n=1}^{\infty}\left\{\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.\xi\right].f(0) +\int_0^{\xi}\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.(\xi-\xi')\right].f'(\xi').\mathrm{d}\xi'. \quad (16a)\right.\right.$$

Forming the difference between (16a) and (17) one obtains:

$$\frac{\mathrm{d}\vartheta_w}{\mathrm{d}\xi}-\frac{\mathrm{d}\vartheta_m}{\mathrm{d}\xi} =2.\bar{q}.\frac{2.\bar{L}}{Pe}.\sum_{n=1}^{\infty}\left\{\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.\xi\right].f(0) +\int_0^{\xi}\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.(\xi-\xi')\right].f'(\xi').\mathrm{d}\xi'\right\}. \quad (18)$$

It can be at once noticed from the above relation that the exact fulfilment of equations (2) or (2a) with the varying heat flux along the duct walls ($f(\xi)\not\equiv 1$, $f'(\xi)\not\equiv 0$) could only be possible in the following cases:

(a) $\bar{q}=0$
(b) $\bar{L}=\infty$
(c) $Pe=0$.

Only the case (a) is mathematically sensible, whereas the cases (b) and (c) lead to indefiniteness. From the physical standpoint the cases (a) and (c) have no sense and the case (b) is practically unrealizable. So it results from the above that *equation* (2) *with varying heat flux can never be satisfied exactly*.

Let us investigate now the situation with the constant heat flux along the wall of the finite duct, i.e. the case when $f(\xi)=f(0)\equiv 1$ and $f'(\xi)\equiv 0$. In this case the relation (18) takes the form:

$$\frac{\mathrm{d}\vartheta_w}{\mathrm{d}\xi}-\frac{\mathrm{d}\vartheta_m}{\mathrm{d}\xi}=2.\bar{q}.\frac{2.\bar{L}}{Pe}.\sum_{n=1}^{\infty}\exp\left[-\frac{2.\bar{L}}{Pe}.(n.\pi)^2.\xi\right]. \quad (19)$$

The cases (a) to (c) considered above are also trivial for the constant heat flux at the duct wall. However, some circumstances may take place when one can expect an approximate fulfilment of the assumption (1) as well as its consequences (2) and (3). It can be noticed at once that at the duct inlet (for $\xi = 0$) the equations (2a) or (2) will never be satisfied. Now, let us investigate the situation at the duct outlet ($\xi = 1$). In Table 1 there are listed the calculated values, which appear on the right side of equation (19) for some chosen values of parameters $2\bar{q}$ and $2\bar{L}/Pe$.

Table 1. Values of the difference $(d\vartheta_w/d\xi) - (d\vartheta_m/d\xi)$ for $\xi = 1$

	$2\bar{L}/Pe$		
$2\bar{q}$	0·1	1·0	10·0
0·1	$3{\cdot}91 . 10^{-2}$	$5 . 10^{-6}$	$1{\cdot}26 . 10^{-44}$
1·0	$3{\cdot}91 . 10^{-1}$	$5 . 10^{-5}$	$1{\cdot}26 . 10^{-43}$
10·0	$3{\cdot}91 . 10^{0}$	$5 . 10^{-4}$	$1{\cdot}26 . 10^{-42}$
10^{+5}	$3{\cdot}91 . 10^{+4}$	$5 . 10^{0}$	$1{\cdot}26 . 10^{-38}$

An analysis of the above values allows to state that *a good approximation of equation* (2) (*or the admissibility of the assumption* α = *const*) *may be expected only for the moderate values of the parameter* $2\bar{q}$ *and for* $2\bar{L}/Pe \geqslant 1$.

However, one can imagine such values of the parameter $2\bar{q}$, that even for the relatively long ducts (i.e. for $2\bar{L}/Pe \gg 1$) the RHS of equation (2) will be much greater than zero. So, it results from the above that in such cases the assumption α = const is not admissible. The assumption α = const seems to be admissible for the ranges $0 < 2\bar{q} \leqslant 10{\cdot}0$ and $2\bar{L}/Pe \geqslant 1$. These ranges seem to be valid also for other duct geometries as well as for the laminar and turbulent flow, because in such cases the eigenvalues are greater than the successive multiples of π-number, [23] and [24].

The above considerations have shown that not only the constant heat flux at the duct wall and the fully developed flow (i.e. the invariable velocity profile) are sufficient conditions for the admissibility of the assumption α = const and for all consequences resulting from it.

It seems to be useful to supplement the assertion contained in [2] by the restriction expressed through the ranges of parameters $2\bar{q}$ and $2\bar{L}/Pe$.

The considerations presented above constitute further confirmation of the assertion by Luikov *et al.* expressed in [25] that in numerous cases the mixed boundary condition takes no physical sense. This is, of course, a case when α may not be treated as constant. The solution process of the heat-transfer problem must then be performed accordingly to the principles appropriate to conjugated problems presented in [22] and [25].

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QUAND PEUT-ON TRAITER LE COEFFICIENT DE TRANSMISSION DE LA CHALEUR α COMME UNE CONSTANTE

Résumé—On discute les conditions sous lesquelles le coefficient de transmission de la chaleur α peut être considéré comme constant. A l'aide de l'exemple de l'écoulement piston dans un canal plan de longueur finie avec flux thermique imposé à la paroi, un critère a été obtenu pour ce type de problème et le domaine des valeurs numériques possibles est déterminé.

WANN DARF MAN MIT KONSTANTEN WÄRMEÜBERGANGSKOEFFIZIENTEN RECHNEN?

Zusammenfassung—Es wird die Frage behandelt, wann die Annahme eines konstanten Wärmeübergangskoeffizienten α bei erzwungener Konvektion zulässig ist. Dazu wird die Kolbenströmung betrachtet. Am Beispiel eines endlichen, ebenen Kanals mit vorgegebener Wärmestromdichte an der Wand werden die notwendigen Kriterien und der Bereich ihrer numerischen Werte festgelegt.

КОГДА ДОПУСКАЕТСЯ РАССМАТРИВАТЬ КОЭФФИЦИЕНТ ТЕПЛООТДАЧИ α ПОСТОЯННЫМ?

Аннотация — Обсуждается проблема, когда допускается рассматривать коэффициент теплоотдачи α как постоянную величину. На примере стержневого течения в конечном (в направлении течения) канале между тонкими плитами с неоднородным граничным условием второго рода выведены критерияльные параметры а также определены интервалы их численных значений.